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THE USE OF MONTHLY AND QUARTERLY DATA IN AN ARMA MODEL

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This paper deals with specification, prediction and length of interval between the observations in an ARMA model. An AR(1) model is found to be suitable for a specific monthly time series. From this series we construct two types of quarterly series and derive the corresponding ARMA models. The theoretical parameter values of the quarterly models, given the monthly model, are compared with the values found empirically when no monthly series exists. By using the variance of the predictor error, we assess the performance of all specifications in predicting up to one year ahead. We show that while the monthly model performs best in theory, the values computed directly from the estimates prove in our empirical example the quarterly models to be preferable in most cases where we are to predict more than one quarter ahead.

1. Introduction

The specification of a discrete stochastic process depends on the length of the time interval between the observations. The availability of the data usually determines whether we use yearly, quarterly or even monthly observations for our model. In general, we will choose the shortest interval possible, since it provides the greatest number of observations and hence the most detailed information about the process. A longer interval may, however, be preferable, when predictions for many periods ahead are to be made, and we do not want our model to account for irrelevant shocks and movements in the short run. On the other hand, we may need to know whether the loss of information seriously affects our results when, for instance, we are forced to use quarterly instead of monthly data, when the latter either do not exist or are very difficult to obtain.

For discrete stationary stochastic processes [ARMA models, using the Box and Jenkins (1970) terminology] some results concerning the relationship between specification and length of interval have been established. Telser (1967), Brewer (1973), and Amemiya and Wu (1972) provided the specification of the

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model with an interval of length m (m integer) when the specification for length 1 is given. Telser investigated the possibility of interpolating missing observations, Brewer considered both ARMA models and transfer functions, and Amemiya and Wu examined the performance of various predictors when data are aggregated over a longer interval.

The effects of time aggregation in a larger econometric model with exogenous variables were explored by Engle and Liu (1972). They found that the theoretical results for the estimation bias of the distributed lag parameters were confirmed empirically when comparing the estimates of Liu's monthly model of the U.S. economy with those of the quarterly and yearly form of that model.

The present paper, to some extent, links up with the studies of Amemiya and Wu and Engle and Liu. For a series of monthly data, the appropriate ARMA specification is identified as AR(1) and its parameters are estimated. From the monthly data we construct two alternative types of quarterly series:

(i) by taking every third observation (series I–III),
(ii) by taking the average of the monthly figures in each quarter (series of averages).

For these series we calculate the parameter values of the specifications, which correspond to the AR(1) monthly model and compare them with those values that should hold in theory. We also consider the specification which follows from direct identification of the series of averages, and which is, given the monthly model, a misspecification. Thereafter the performance of all specifications in predicting up to one year ahead is assessed by calculating the variances of the predictor error. For the quarterly models we again compute the values that should hold in theory – should the series in fact be generated by the monthly AR(1) model, but should no monthly data exist – and the values that are found empirically. Finally some conclusions are drawn. Most of the algebra is to be found in appendix B.

2. The monthly model

This paper investigates the series of the difference between the yield on mortgages and the yield on government loans (RH − RO), where

RH: yield on mortgage loans granted by mortgage banks against first mortgages,
RO: yield on the latest three long-term government loans.

The reference period is January 1961 – March 1974 providing 159 monthly and 53 quarterly observations. The data are given in appendix A.

Our main reason for choosing this series is that a suitable (quarterly) ARMA model for it is already being applied by us in a study of the mortgage market. Our sole purpose is to show, by means of a numerical example, some aspects of the relationship between a monthly and a quarterly model. The model for this series has the attractive property of being simple and stable.

The most likely specification for the monthly series, as identified both by sample autocorrelations and sample partial autocorrelations, seemed to be the AR(1) model with a constant term,[1]

$$(z_t^0-\mu) = \phi(z_{t-1}^0-\mu)+\boldsymbol{a}_t, \tag{1}$$

where $\boldsymbol{a}_t$ represents white noise with variance σ_a^2 and t is the monthly index $(t = 1, \ldots, T)$.

In our computer program, based on Box and Jenkins (1970), Program 3, though without back forecasts,[2] the estimators $\hat{\boldsymbol{\phi}}$ of ϕ and $\hat{\boldsymbol{\mu}}$ of μ are identical to the ordinary least squares estimators.

The results are

$$\hat{\mu} = 1.000, \qquad \text{var}\,\hat{\boldsymbol{a}}_t = 0.0246,$$

$$\hat{\phi} = 0.841, \qquad \text{var}\,\hat{\boldsymbol{\phi}} = 0.00182.$$

To test whether the residuals show a systematic pattern, we applied the portmanteau lack of fit test [see Box and Jenkins (1970, pp. 290–293)]. This statistic is approximately distributed χ^2 and the value of 43.4 with 46 degrees of freedom does not reject the null-hypothesis that the residuals are white noise.

Bartlett's (1946) formula provides another diagnostic check – in the AR(1) model holds[3]

$$\text{var}\,\hat{\boldsymbol{\phi}} \simeq \frac{1-\phi^2}{T+1}\,. \tag{2}$$

When we substitute the estimate 0.841 for ϕ, the theoretical var $\hat{\boldsymbol{\phi}}$ of this model should be 0.00184, which almost equals 0.00182 calculated directly from the data. These checks may indicate that the AR(1) model is an adequate specification for the monthly series.

Formula (2) does not account for the constant term μ. We shall omit this term in the rest of our study since it simplifies the calculations and affects the results very little.[4] Accordingly we rewrite (1) as

$$z_t = \phi z_{t-1}+\boldsymbol{a}_t, \tag{3}$$

where

$$z_t = z_t^0-\hat{\mu}.$$

[1]Random variables are displayed in bold type.

[2]Miss Volgenant and Mr. Van de Gevel adapted the Box and Jenkins programs to our computer.

[3]All approximative formulae of this paper are of $O(T^{-2})$. These remainders are neglected in the calculations.

[4]Thus, the mean of the series is always considered as given and as equal to its value as calculated in fact (which is invariably 1.0).

3. The quarterly models

Three different quarterly series are obtained from the monthly series by taking every third observation. Thus, series I consists of the observations for the first month of each quarter, series II of the observations for the second month and series III of the observations for the third month of each quarter.

Repeated substitution of the AR(1) model (3),

$$z_t = \phi^3 z_{t-3} + \phi^2 \boldsymbol{a}_{t-2} + \phi \boldsymbol{a}_{t-1} + \boldsymbol{a}_t, \tag{4}$$

shows that, given the monthly model, series I–III may all be represented theoretically by the same process:

$$z_\tau^* = \phi_* z_{\tau-1} + \boldsymbol{a}_\tau^*, \tag{5}$$

where

$$z_\tau^* = z_{(\tau-1)3+j}, \qquad j = \begin{cases} 1, & \text{for series I,} \\ 2, & \text{for series II,} \\ 3, & \text{for series III,} \end{cases}$$

and

$$\phi_* = \phi^3,$$

$$\operatorname{var} \boldsymbol{a}_\tau^* = (\phi^4 + \phi^2 + 1) \operatorname{var} \boldsymbol{a}_t .$$

Since cov $(\boldsymbol{a}_\tau^*, \boldsymbol{a}_{\tau+i}^*) = 0$ for all $i \neq 0$, $\boldsymbol{a}_\tau$ is white noise and accordingly (5) represents an AR(1) model.

The quarterly series of averages is formed by taking the averages of the monthly figures in each quarter. When the AR(1) model (3) is valid for the monthly series, it can be proved[5] that this quarterly series, defined as

$$z_\tau^+ = \tfrac{1}{3}\left(z_{(\tau-1)3+1} + z_{(\tau-1)3+2} + z_{(\tau-1)3+3}\right),$$

obeys in theory the ARMA(1, 1) process

$$z_\tau^+ = \phi_+ z_{\tau-1}^+ + \boldsymbol{a}_\tau^+ - \theta \boldsymbol{a}_{\tau-1}^+ . \tag{6}$$

In order to examine the model we would have arrived at if the monthly series either did not exist or were disregarded we applied the procedure of identification for this series of averages. From the sample autocorrelations and sample partial autocorrelations the AR(1) specification again appeared to be the most appropriate; no evidence for an ARMA(1, 1) model emerged.

[5]All subsequent proofs and algebra are given in appendix B.

This AR(1) model,

$$z_\tau^+ = \phi_\times z_{\tau-1}^+ + \boldsymbol{a}_\tau^\times , \tag{7}$$

is, in theory, a misspecification for the series of averages when we accept model (3) as the correct specification for the original monthly series. Then $\boldsymbol{a}_\tau^\times$ is no white noise.

Table 1 shows the parameter values of the quarterly models. The table is divided into two parts: on the left-hand side the values are given which should hold when the monthly series is in fact produced by the AR(1) model with parameters as estimated. The results for the parameters and the variance of the noise are obtained directly as functions of $\hat{\phi}$ and var $\hat{\boldsymbol{a}}_t$. However, for the variance of the parameters the theoretical values are calculated should the parameters be estimated using artificial quarterly data generated by the monthly model. In a simulation study they would be the mean values of the estimates of the variance of the parameters resulting from the experiments (neglecting the bias of the approximative formulae). The values found empirically using the historical quarterly data appear on the right-hand side of table 1.

The theoretical values of the misspecified model (7) are calculated for the AR(1) model that has the smallest var $\boldsymbol{a}_\tau^\times$ and therefore corresponds best to the ARMA(1, 1) model which is the 'correct' specification for this series.

Table 1

Parameter values of the quarterly models.

	Theoretical values			Estimates			
							portmanteau test-statistic (χ^2 distributed)
AR(1) model	ϕ_*	var $\boldsymbol{a}_\tau^*$	var $\hat{\phi}_*$	$\hat{\phi}_*$	var $\hat{\boldsymbol{a}}_\tau^*$	var $\hat{\phi}_*$	34 degrees of freedom
Series I				0.611	0.0530	0.0123	32.0
Series II	0.595	0.0543	0.0122	0.605	0.0502	0.0123	14.7
Series III				0.552	0.0683	0.0137	18.0
ARMA(1, 1) model							
Series of average	ϕ_+	var $\boldsymbol{a}_\tau^+$	var $\hat{\phi}_+$	$\hat{\phi}_+$	var $\hat{\boldsymbol{a}}_\tau^+$	var $\hat{\phi}_+$	33 degrees of freedom
	0.595	0.0360	0.0236	0.707	0.0316	0.0167	20.8
	θ	var $\hat{\theta}$		$\hat{\theta}$	var $\hat{\theta}$		
	−0.217	0.0348		−0.109	0.0346		
AR(1) model							
Series of averages	$\phi_\times$	var $\boldsymbol{a}_\tau^\times$	var $\hat{\phi}_\times$	$\hat{\phi}_\times$	var $\hat{\boldsymbol{a}}_\tau^\times$	var $\hat{\phi}_\times$	34 degrees of freedom
	0.702	0.0368	0.0074	0.751	0.0311	0.0084	22.1 (!)

The difference between the theoretical values and the estimates appears to be small for series I–III, especially for I and II. This may be regarded as another indication for the adequacy of the AR(1) specification for the monthly series. Larger, though not significant, differences appear in the ARMA(1, 1) model for the series of averages. Here a part of the moving average effect is ascribed in practice to autoregression. In no case did the portmanteau test-statistic suggest model-inadequacy, not even in the case of the misspecified AR(1) model. It is noticeable that for all series, except for series III, the theoretical variance of the residuals is larger than the estimated variance.

4. Prediction

In order to compare the predictive performance of the various models we shall compute the variances of the predictor errors[6] when predicting up to one year ahead. This variance can be split up into two parts:

(1) the autonomous part of the variance, caused by the disturbances after the last observation (month T or quarter Y): VARAUT;
(2) the part of the variance caused by the estimation errors of the parameters: VARPAR.

Calculation of VARAUT is generally easy but VARPAR causes complications which necessitate simplifying assumptions. As far as possible we shall provide fair approximations for VARPAR but in the case of the ARMA(1, 1) model no satisfactory solution could be found. Since VARPAR approaches to zero for a large number of observations, this part of the variance is usually neglected [e.g., Amemiya and Wu (1972)], while Box and Jenkins (1970) mention it in their Appendix A.7.3 only. Yet VARPAR may be important in the case of a limited number of observations, especially when the predictive performances of monthly and quarterly models are to be compared. Namely, the same reference period comprises three times as many monthly observations than quarterly observations.

Table 2 reports the results for the monthly model. Since we are concerned mainly with predicting a number of periods ahead and with comparing the performance of this model with the quarterly models, the table shows only the outcomes for quarters. The variance of the predictor error for series I–III depends on the last month of observation. Table 2 only shows the maximum values, namely when a multiple of three months ahead is to be predicted. For the series of averages the predictions are computed by averaging the predictions

[6]In the models considered in this paper the variance of the predictor error may discriminate the predictive performance of the various models in a better way than comparison of ex ante predictions and realisations. E.g., in the last quarters and months of our sample-period the values of the series are close to one, i.e., the mean of the series. In that case the predictions resulting from all models of this paper are broadly the same.

Table 2

Variance of the predictor error of the monthly AR(1) model.

	Quarter	VARAUT	VARPAR	Total variance
Series I–III	I	0.0543	0.0007	0.0550
	II	0.0735	0.0010	0.0745
	III	0.0803	0.0008	0.0811
	IV	0.0827	0.0005	0.0832
Series of averages	I	0.0297	0.0004	0.0301
	II	0.0574	0.0009	0.0584
	III	0.0673	0.0009	0.0681
	IV	0.0707	0.0006	0.0713

for the three relevant months. For this series the values of the variance of the predictor error are given which hold at the end of each quarter.

Table 3 gives the outcomes for the quarterly models. Here the same line is followed as in table 1 with the parameter values: the theoretical values appear on the left-hand side, while the values that are directly calculated from the estimates of the quarterly models, are given on the right-hand side. The calculation of the theoretical values of VARPAR again assumes that the theoretical parameter estimates are obtained from artificial quarterly series generated by the monthly AR(1) model.

Comparison of table 2 and table 3 shows that for the series I–III the theoretical VARAUT of the monthly and quarterly models are equal when a multiple of three months ahead is to be predicted, but the VARPAR of the latter model, and hence its total variance, is always larger. This is generally valid. For the series of averages the theoretical outcomes also conform to our a priori expectations. The monthly model performs best since it uses the most detailed information. The ARMA(1, 1) quarterly model (comparing the VARAUT) is second best, and the AR(1) model yields the worst predictions since it is a misspecification. The differences, however, are small.

When we compare the values calculated directly from the estimates of the models, the picture changes for the series of averages. Now, except for the first quarter, the AR(1) quarterly model performs best. In fact we should not consider this model a misidentification. From identification of the series it was shown to be the most likely specification and after estimation of the parameters, the portmanteau test-statistic showed no further motive to look for a systematic pattern of the residuals. Moreover, the predictive performance appears to be slightly better than in the properly specified ARMA(1, 1) model.

In reality time series are formed in a very complicated way and never generated by simple stochastic processes, as they are in a Monte-Carlo experiment. Consequently this is true for the monthly series which, moreover, are also

Table 3
Variance of the predictor error of the quarterly models.

		Theoretical values			Estimates		
	Quarter	VARAUT	VARPAR	Total variance	VARAUT	VARPAR	Total variance
AR(1) model							
Series I	I				0.0530	0.0010	0.0541
	II				0.0728	0.0016	0.0744
	III				0.0802	0.0013	0.0815
	IV				0.0829	0.0009	0.0838
Series II	I	0.0543	0.0010	0.0553	0.0502	0.0010	0.0511
	II	0.0735	0.0015	0.0749	0.0685	0.0014	0.0700
	III	0.0803	0.0012	0.0815	0.0753	0.0012	0.0764
	IV	0.0827	0.0007	0.0834	0.0777	0.0008	0.0785
Series III	I				0.0683	0.0013	0.0696
	II				0.0891	0.0016	0.0907
	III				0.0954	0.0011	0.0965
	IV				0.0973	0.0006	0.9079
ARMA(1,1) model							
Series of averages	I	0.0360			0.0316		
	II	0.0596			0.0526		
	III	0.0680			0.0631		
	IV	0.0710			0.0683		
AR(1) model							
Series of averages	I	0.0368	0.0005	0.0373	0.0311	0.0006	0.0317
	II	0.0604	0.0011	0.0614	0.0486	0.0013	0.0499
	III	0.0688	0.0012	0.0700	0.0585	0.0017	0.0602
	IV	0.0717	0.0010	0.0727	0.0640	0.0017	0.0657

composed of averages and therefore cannot possibly be produced by an AR(1) process. Yet, in time-series analysis we do not pursue perfect specifications but only try to find adequate models. Given this aim, these simple AR(1) models provide us with a satisfactory instrument for description and prediction of both the monthly and quarterly time series, even when, as in the present case, they conflict theoretically. This example emphasizes the validity of Box and Jenkins' principle of parsimony.

It is noticeable that both the AR(1) and the ARMA(1, 1) quarterly models for the series of averages perform better than the monthly model when half a year or a longer period is to be predicted ahead. This may indicate that the monthly model does, indeed, account for movements and shocks which are irrelevant for a longer term.

Since a relatively large number of observations are used for estimating the parameters, VARPAR is small in comparison with the total variance of the predictor error in all models of this paper. In the case of, say, more than 40 observations no serious downward bias is caused by the usual omission of this part of the variance. We have calculated what the percentage share of VARPAR in the total variance of the predictor error would have been, if a monthly figure were predicted one year ahead by means of the monthly AR(1) model – 0.6 %, the quarterly AR(1) model – 0.9 %, or a corresponding yearly AR(1) model – 7.1 %.

So, if we had examined the relationship between quarterly and yearly models in the same reference period, VARPAR would have been of greater importance, especially when we are to compare the predictive performance of these models. In that case a considerable loss of information would have occurred by switching to a longer interval.

5. Conclusion

This paper compares an AR(1) monthly model with the corresponding AR(1) and ARMA(1, 1) quarterly models for a specific time series. Our main purpose is to illustrate, by means of a numerical example, the transition to a longer interval between observations. We appreciate that the examination of only one series seriously restricts the range of our conclusions. Yet, some of them may have a general validity.

Firstly, the relationship between specification and length of interval provides a (global) diagnostic check: an indication for adequacy of the models can be obtained by comparing the actual estimates of the quarterly models with the theoretical values, given the monthly model. Secondly, if the observations are really generated by the monthly model, it is best to use this model for prediction. This corresponds to our a priori expectations. Yet the differences with the quarterly models are small.

From the actual estimates of the models it follows, however, that (in this particular case) the AR(1) quarterly model yields the best predictions for the series of averages when more than one quarter is to be predicted ahead. This model appears to be slightly superior to both the AR(1) monthly model and the ARMA(1, 1) model which is in theory the correct quarterly specification. Since no test-statistic did indicate the inadequacy of the AR(1) model for the series of averages, we should be wary of calling it a misidentification albeit a misspecification given the monthly AR(1) model.

The example in this paper shows the importance of parsimony for the identification of an ARMA model. Moreover, we demonstrated that we do not invariably need to use the data available for the shortest interval between the observations when we are to predict many periods ahead. It would be interesting to investigate whether this last conclusion is also valid in large econometric models.

Appendix A

Table 4

Monthly figures of (RH−RO); RH = yield on mortgage loans granted by mortgage banks against first mortgages, RO = yield on the latest three long-term government loans.[a]

	1961	1963	1965	1967	1969	1971	1973
January	0.66	0.78	1.34	1.30	0.78	1.70	1.04
February	0.70	1.00	1.37	0.97	0.57	1.43	1.10
March	0.74	1.05	1.13	0.96	0.41	1.44	1.10
April	0.63	1.09	1.04	0.80	0.61	1.37	1.09
May	0.70	1.05	0.92	0.62	0.85	1.20	1.05
June	0.66	0.75	1.15	0.51	0.85	1.19	0.70
July	0.61	0.73	0.99	0.56	1.11	1.39	0.88
August	0.52	0.77	1.32	0.84	1.05	1.41	0.81
September	0.60	0.77	1.46	0.87	0.96	1.40	1.08
October	0.61	0.84	1.24	0.87	1.31	1.39	1.39
November	0.70	0.66	1.01	0.76	1.49	1.62	1.16
December	1.10	0.68	1.04	0.86	1.35	1.59	0.49
	1962	**1964**	**1966**	**1968**	**1970**	**1972**	**1974**
January	1.17	0.67	1.08	0.81	1.32	1.36	0.74
February	1.23	0.56	0.94	0.77	1.24	1.31	0.90
March	0.85	0.62	0.81	0.74	1.47	0.99	0.91
April	0.78	0.73	1.00	0.80	1.32	0.89	
May	0.71	0.70	0.98	0.78	1.23	0.87	
June	0.55	0.74	1.02	0.72	1.33	0.94	
July	0.56	0.93	1.16	0.66	1.48	1.03	
August	0.74	1.00	0.96	0.92	1.49	1.27	
September	0.80	1.50	1.23	0.99	1.48	1.20	
October	0.75	1.30	1.10	0.98	1.49	1.10	
November	0.74	1.18	1.02	0.70	1.55	0.93	
December	0.79	1.15	1.08	0.65	1.73	1.00	

[a] *Source:* Quarterly Statistics, De Nederlandsche Bank N.V.

Appendix B

The ARMA (1, 1) quarterly model for the series of averages

From repeated substitution of the monthly AR(1) model,

$$z_t = \phi z_{t-1} + a_t, \tag{B.1}$$

and adding, it follows that the quarterly series of averages z_τ^+ obeys the process

$$z_\tau^+ = \phi_+ z_{\tau-1}^+ + b_\tau^+, \tag{B.2}$$

where $\phi_+ = \phi^3$ and $\boldsymbol{b}_\tau^+$ is the MA(1) process

$$\boldsymbol{b}_\tau^+ = \boldsymbol{a}_\tau^+ - \theta \boldsymbol{a}_{\tau-1}^+, \tag{B.3}$$

with the first autocorrelation ρ_1

$$\rho_1 = \frac{\phi + 2\phi^2 + \phi^3}{3 + 4\phi + 5\phi^2 + 4\phi^3 + 3\phi^4},$$

and θ that root of

$$\theta^2 + \theta/\rho_1 + 1 = 0,$$

which allows (B.3) to be an invertible process [i.e., $|\theta| < 1$, see Box and Jenkins (1970, p. 69)]. Consequently z_τ^+ obeys the ARMA(1, 1) process

$$z_\tau^+ = \phi_+ z_{\tau-1}^+ + \boldsymbol{a}_\tau^+ - \theta \boldsymbol{a}_{\tau-1}^+. \tag{B.4}$$

The theoretical values of var $\hat{\boldsymbol{\phi}}_+$ and var $\hat{\boldsymbol{\theta}}$ are calculated from (7.2.23) of Box and Jenkins, while from (B.3)

$$\text{var } \boldsymbol{a}_\tau^+ = \frac{\text{var } \boldsymbol{b}_\tau^+}{1+\theta^2} = \frac{\frac{1}{9}(3 + 4\phi + 5\phi^2 + 4\phi^3 + 3\phi^4) \text{ var } \boldsymbol{a}_t}{1+\theta^2}.$$

The AR(1) quarterly model for the series of averages

The (misspecified) AR(1) model for the series of averages z_τ^+,

$$z_\tau^+ = \phi_\times z_{\tau-1}^+ + \boldsymbol{a}_\tau^\times, \tag{B.5}$$

corresponds best to the ARMA(1, 1) model (B.4), and thus to the monthly AR(1) model (B.1) when var $\boldsymbol{a}_\tau^\times$ is minimized with respect to $\phi_\times$. Var $\boldsymbol{a}_\tau^\times$ reaches its minimum when $\phi_\times$ is equal to the first autocorrelation of the ARMA (1, 1) model (B.4):

$$\phi_\times = \frac{(1-\phi_+\theta)(\phi_+ - \theta)}{1+\theta^2 - 2\phi_+\theta}.$$

In fact $\boldsymbol{a}_\tau^\times$ in (B.5) is no white noise but obeys the ARMA(1, 2) model (B is the lag-operator):

$$(1-\phi_+ B)\boldsymbol{a}_\tau^\times = (1-\phi_\times B)(1-\theta B)\boldsymbol{a}_\tau^+. \tag{B.6}$$

Var $\boldsymbol{a}_\tau^\times$ can be computed using this formula.

In order to calculate the theoretical value of var $\hat{\phi}_\times$ we use Bartlett's (1946) formula,

$$\text{var}\,\hat{\phi}_\times \simeq \frac{1}{Y+1}\sum_{i=-\infty}^{\infty}\{\rho_i^2+\rho_{i-1}\rho_{i+1}-4\rho_1\rho_i\rho_{i+1}+2\rho_i^2\rho_1^2\}\,,$$

where $\hat{\phi}_\times$ is the OLS-estimator of z_τ^+ on $z_{\tau-1}^+$. By substituting the autocorrelations of the ARMA(1, 1) model into this formula, it follows that

$$\text{var}\,\hat{\phi}_\times \simeq \frac{1}{Y+1}\left\{1+2\phi_\times\phi_+-7\phi_\times^2+\frac{4\phi_\times^2(1+\phi_\times^2-2\phi_\times\phi_+)}{1-\phi_+^2}\right\}.^{7} \qquad \text{(B.7)}$$

Prediction by means of the monthly model

In the AR(1) model (B.1) the mean square error predictor for z_{T+l} is

$$\hat{z}_{T,l} = \hat{\phi}^l z_T,$$

where l stands for the number of months to be predicted ahead. The predictor error is defined as

$$e_{T,l} = z_{T+l}-\hat{z}_{T,l},$$

and its variance is

$$\text{var}\,(e_{T,l}) = \text{var}\,\{z_T(\hat{\phi}^l-\hat{\phi}^l)\}+\text{VARAUT}, \qquad \text{(B.8)}$$

where

$$\text{VARAUT} = \frac{(1-\phi^{2l})}{1-\phi^2}\,\sigma_a^2 \qquad \text{(B.9)}$$

represents the autonomous part of the variance, caused by the disturbances after the last observation T.

To evaluate VARPAR = var $\{z_T(\phi^l-\hat{\phi}^l)\}$ – i.e., the part of variance caused by the estimation error of the parameter – we assume by way of simplification that $\hat{\phi}$ and z_T are stochastic independent. Then [see Box and Jenkins (1970, Appendix A7.3)]

$$\text{VARPAR} \simeq \frac{l^2\phi^{2l-2}}{1-\phi^2}\,\sigma_a^2\,\text{var}\,(\hat{\phi})\,. \qquad \text{(B.10)}$$

[7]If $\phi_+ = \phi_\times$ (the AR(1) model) we obtain formula (2).

The variance of the predictor error when predicting the series I–III a multiple of three months ahead by means of the monthly model is computed by substitution of $l = 3\kappa$ in the above formulae, where κ is the number of quarters to be predicted ahead.

In the case of the series of averages the predictor error may be written

$$F_{T,\kappa} = \tfrac{1}{3}\,(e_{T,l-2}+e_{T,l-1}+e_{T,l})\,.$$

Again assuming z_T and $\hat{\phi}$ to be stochastic independent we calculated that

$$\operatorname{var} F_{T,\kappa} = \text{VARAUT}+\text{VARPAR},$$

where

$$\text{VARAUT} = \frac{1}{9}\frac{\sigma_a^2}{1-\phi^2}\,\{3-\phi^{2l-4}(1+\phi^2+\phi^4) + 2\phi(2-\phi^{2l-4}-\phi^{2l-2})+2\phi^2(1-\phi^{2l-4})\},$$

and

$$\text{VARPAR} \simeq \frac{1}{9}\frac{\sigma_a^2}{1-\phi^2}\,\{[(l-2)\phi^{l-3}+(l-1)\phi^{l-2}+l\phi^{l-1}]^2\}\operatorname{var}\hat{\phi}\,.$$

Prediction by means of the quarterly models

When predicting series I–III with the quarterly AR(1) model (5), the values of VARAUT and VARPAR are computed by substitution of ϕ_*, σ_{a*}^2 and κ for ϕ, σ_a^2 and l in (B.9) and (B.10).

In the case of predicting the quarterly series of averages by the ARMA (1, 1) quarterly model (B.4) we can, once again, split the variance of the predictor error into VARAUT and VARPAR where

$$\text{VARAUT} = \sigma_{a+}^2\left\{1+(\phi_+-\theta)^2\frac{(1-\phi_+^{2\kappa-2})}{1-\phi_+^2}\right\},$$

but VARPAR is too complicated to evaluate and therefore no values are given for it in table 3.

When we are to calculate the variance of the predictor error that holds in theory in the case of the AR(1) quarterly model (B.5) we should take the misspecification into account. In this case the variance of the predictor error cannot be divided into two independent parts. However, the covariance between VARAUT and VARPAR will be small and since evaluation of this term is too complicated, we neglect it in our calculations.

In this model,

$$\text{VARAUT} = \left\{\frac{1-\phi_{\times}^{2\kappa}}{1-\phi_{\times}^{2}} \operatorname{var} \boldsymbol{a}_{\tau}^{\times} + 2\sum_{i=1}^{\kappa-1} \phi_{\times}^{i} \frac{1-\phi_{\times}^{2(\kappa-i)}}{1-\phi_{\times}^{2}} \operatorname{cov}(\boldsymbol{a}_{\tau}^{\times}, \boldsymbol{a}_{\tau-i}^{\times})\right\},$$

where $\operatorname{var} \boldsymbol{a}_{\tau}^{\times}$ and $\operatorname{cov}(\boldsymbol{a}_{\tau}^{\times}, \boldsymbol{a}_{\tau-i}^{\times})(i = 1, 2, 3)$ can be expressed in terms of $\phi_{\times}$, ϕ_{+}, θ and σ_{a+}^{2} from the ARMA (1, 2) model (B.6) of $\boldsymbol{a}_{\tau}^{\times}$. Since

$$E\hat{\phi}_{\times} = \phi_{\times} + O(Y^{-1})$$

– see Kendall and Stuart (1968, ch. 48) – we can derive that, on the analogy of the correctly specified AR(1) model,

$$\text{VARPAR} \simeq \frac{\kappa^{2}\phi_{\times}^{2(\kappa-1)}}{1-\phi_{\times}^{2}} \sigma_{a\times}^{2} \operatorname{var} \hat{\phi}_{\times},$$

where $\operatorname{var} \hat{\phi}_{\times}$ is given by (B.7).

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